HEAT TRANSFER BETWEEN A FLAT PLATE WITH A NONSTATIONARY SURFACE TEMPERATURE AND A LAMINAR FLOW OF LIQUID AROUND IT

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Analytical formulas are derived to describe the nonstationary heat transfer taking place when a laminar flow of liquid passes around a flat plate, the surface temperature being an arbitrary but specified function of time.

The problem of nonstationary convective heat transfer is still a long way from the stage of development which has been achieved, for example, in the theory of nonstationary heat conduction [1]. Recently there have been a number of papers aimed at solving the equation of nonstationary heat transfer in the case of laminar and turbulent flows bounded by a cylindrical surface or parallel flat plates. There have also been papers concerned with heat transfer between a solid and a flow of liquid passing around it [2-6]. These papers have set the stage for a study of nonstationary heat transfer with forced convection.

The aim of the present analysis is one of finding a description for heat transfer between a flat plate with a nonstationary surface temperature and a liquid flowing around it. A flow of incompressible liquid with constant thermophysical properties flows around a semiinfinite flat plate. The motion of the liquid is stationary (steady-state) and its temperature is constant. One of the three following cases may apply.

The first case is that in which the surface temperature of the plate changes sharply, i.e., the surface temperature is initially equal to that of the flow of liquid T_∞, then at a certain instant the surface temperature changes instantaneously and is kept constant at a level of T_W. Neglecting energy dissipation and changes in the thermophysical properties of the incompressible liquid, the energy-transfer equation may be converted into the form

$$\frac{\partial \vartheta}{\partial \tau} + u\,\frac{\partial \vartheta}{\partial x} + v\,\frac{\partial \vartheta}{\partial y} = a\,\frac{\partial^2 \vartheta}{\partial y^2}. \tag{1}$$

By analogy with [8], when Pr = 1 we may rewrite Eq. (1) in the following way

$$\frac{\partial \vartheta}{\partial \tau} + \varepsilon u\,\frac{\partial \vartheta}{\partial x} = a\,\frac{\partial \vartheta}{\partial y^2}, \tag{2}$$

where

$$\varepsilon = 1 + v\,\frac{\partial \vartheta}{\partial y}\Big/ u\,\frac{\partial \vartheta}{\partial x}.$$

We may replace the velocity component u in (2) by its limiting value U at the outer edge of the boundary layer. The error introduced by this approximation will subsequently be taken into the correction ε; then (2) may be rewritten in the form

$$\frac{\partial \vartheta}{\partial \tau} + \varepsilon U\,\frac{\partial \vartheta}{\partial x} = a\,\frac{\partial^2 \vartheta}{\partial y^2}. \tag{3}$$

Let us introduce the thermal boundary layer $\delta(x, \tau)$, the thickness of which has to be determined, for which purpose we integrate Eq. (3) from y = 0 to y = δ. In integrated form Eq. (3) may be written thus:

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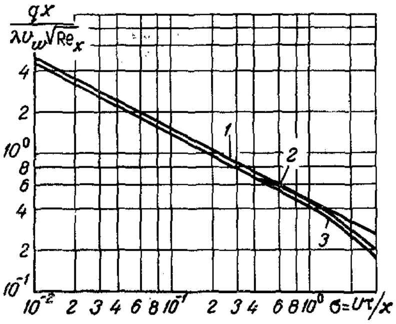

Fig. 1

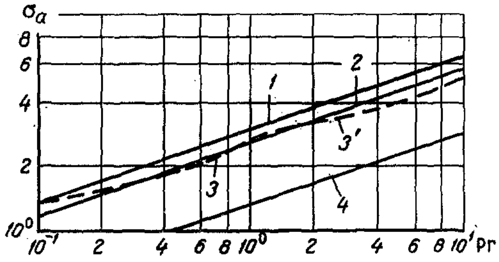

Fig. 2

Fig. 1. Heat flow as a function of time: 1) according to [2]; 2) by Eq. (11a); 3) according to [3].

Fig. 2. Transition time σ_a as a function of the Prandtl number Pr: 1) according to [2]; 2) from Eq. (15); 3,3') according to [7]; 4) according to [4].

$$\frac{\partial}{\partial \tau}\int_0^\delta \vartheta dy + \frac{\partial}{\partial x}\varepsilon U \int_0^\delta \vartheta dy = -a\left(\frac{\partial \vartheta}{\partial y}\right)_{y=0}. \tag{4}$$

In order to carry out the integration, we must make an assumption regarding the profile of the temperature distribution in the boundary layer. To a reasonable approximation we may take a cubic temperature profile

$$\frac{\vartheta}{\vartheta_w} = 1 - 1.5\left(\frac{y}{\delta}\right) + 0.5\left(\frac{y}{\delta}\right)^3. \tag{5}$$

Substituting (5) into (4) and simplifying, we obtain

$$\delta\frac{\partial \delta}{\partial \tau} + \varepsilon U \delta \frac{\partial \delta}{\partial x} = 4a. \tag{6}$$

In the case of an isothermal state of the flat plate, the boundary and initial conditions take the form

$$\delta(0,\ \tau) = 0; \quad \delta(x,\ 0) = 0. \tag{7}$$

Equation (6) may be solved in partial derivatives either by the method of characteristics (as in [6]) or by the similarity method (as in [3] and [4]), or else by a Laplace transformation.

The solution takes the form

$$\delta = \sqrt{8a\tau}; \quad \tau \leqslant \frac{x}{\varepsilon U}; \tag{8a}$$

$$\delta = \sqrt{8ax/\varepsilon U}; \quad \tau \geqslant \frac{x}{\varepsilon U}. \tag{8b}$$

The flow of heat on the surface of the plate may be found from the Fourier law

$$q = -\lambda\left(\frac{\partial \vartheta}{\partial y}\right)_{y=0}. \tag{9}$$

Using this relationship and Eq. (5) we obtain

$$q = \frac{3}{2}\frac{\lambda \vartheta_w}{\delta}. \tag{10}$$

Combining this result with Eq. (8), we obtain an expression for the heat flow on the surface of the plate

$$\frac{qx}{\vartheta_w \lambda} = \mathrm{Nu}_x = 0.531\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2}\sigma^{-1/2}; \quad \sigma \leqslant \frac{1}{\varepsilon}, \tag{11a}$$

$$\frac{qx}{\vartheta_w \lambda} = \mathrm{Nu}_x = 0.531\,\varepsilon^{1/2}\,\mathrm{Pr}^{1/2}\,\mathrm{Re}_x^{1/2}; \quad \sigma \geqslant \frac{1}{\varepsilon}, \tag{11b}$$

where

$$\sigma = \tau U / x.$$

For short time intervals, the equation for determining the heat flow (11b) agrees closely with the solution of the one-dimensional problem of heat propagation by conduction in a semiinfinite space occupied by a liquid [1], which (in our own notation) is:

$$\frac{qx}{\vartheta_w \lambda} = \frac{1}{\sqrt{\pi}} \mathrm{Pr}^{1/2} \mathrm{Re}_x^{1/2} \sigma^{-1/2}. \tag{12}$$

This result is not unexpected, since in the case of very short time intervals the temperature gradient is only nonzero in the immediate neighborhood of the surface of the plate, at which the velocity components u and v are extremely small, and hence convective effects may be neglected. For short time intervals, also, the heat-flow equation agrees closely with the results of [2, 3].

In Fig. 1 Eq. (11a) is compared with the results of [2] and [3] for the Prandtl number Pr = 0.72.

We see from Fig. 1 that for short time intervals all the equations agree quite closely with each other.

In calculating the correction ε we make use of the solution of the stationary problem with which Eq. (11b) should coincide for heat flow over long periods of time. The local dimensionless stationary heat flow for a flat plate may be found from the exact solution of Schlichting [9]

$$\mathrm{Nu}_x = 0.332\, \mathrm{Pr}^{1/3} \mathrm{Re}_x^{-1/2}. \tag{13}$$

Equating (11b) to (13), we obtain an equation for determining the correction

$$\varepsilon = 0.391\, \mathrm{Pr}^{-1/3}. \tag{14}$$

The equations for the heat flow (11a) and (11b) show that, in the case of a sharp change in temperature on the surface of the plate, the heat flow may be described by the solution to the problem of one-dimensional heat conduction, up to a certain instant of time given by

$$\sigma_a = \frac{1}{\varepsilon} = 2.56\, \mathrm{Pr}^{1/3}, \tag{15}$$

after which the solution corresponding to stationary heat transfer may be used for the heat flow.

We see from Eq. (15) that, the greater the Prandtl number, the longer is the time required to reach the steady state.

Equation (15) for the steady-state transition time agrees reasonably well with the analogous expressions obtained in [2, 6]. The expression for the transition time obtained in [3] differs from (15) in respect to the numerical factor, which is roughly half that given in the latter. The reason for this difference is that a linear velocity-distribution profile was used in [3], leading to a physically impossible result. The same result was obtained in [4] using a linear velocity profile.

For comparison, Fig. 2 shows the transition times given by Eq. (15) and those calculated from the equations of [2, 3, 6].

The second case is a generalization of the first case, in which the surface temperature of the plate is an arbitrary function of time. In other words, the plate and the liquid are initially at a temperature T_∞ and then the surface temperature varies arbitrarily with time in the manner $T_W(\tau)$.

The results obtained for the case in which the surface temperature changes abruptly may be generalized to the case in which the surface temperature is an arbitrary function of time $\vartheta_W = \vartheta_W(\tau)$ for $\tau > 0$, if $\vartheta_W(\tau)$ and the derivative $\vartheta'(\tau)$ are fragmentedly continuous for $\tau > 0$ (Duhamel theorem).

According to the Duhamel theorem, on applying Eq. (11) and considering $\vartheta_W(0) = 0$, the heat flow on the surface of the plate is given by the following relation:

$$\frac{qx}{\lambda} = 0.332\, \mathrm{Pr}^{1/3} \mathrm{Re}_x^{1/2} \int_0^{\sigma} \vartheta_w'(\sigma - t) \left(\frac{t}{\sigma_a}\right)^{-1/2} dt; \quad \sigma \leqslant \sigma_a; \tag{16a}$$

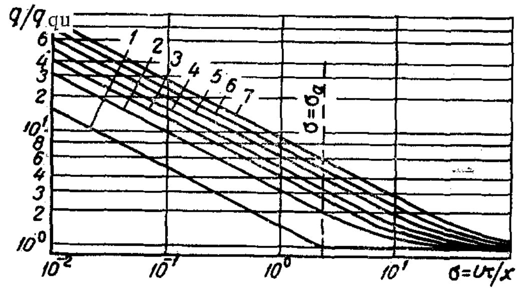

Fig. 3. Dependence of the heat flow on the surface temperature when the latter obeys the law $\vartheta_w = a_n \tau^n$: 1) n = 0; 2) 1; 3) 2; 4) 3; 5) 5; 6) 7; 7) 10.

$$\frac{qx}{\lambda} = 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}\left[\int_0^{\sigma_a} \vartheta_w'(\sigma - t)\left(\frac{t}{\sigma_a}\right)^{-1/2} dt + \vartheta_w(\sigma - \sigma_a)\right];\quad \sigma \geqslant \sigma_a. \tag{16b}$$

In general, if the surface temperature of the plate is a polynomial of the n-th degree in the time, i.e.,

$$\vartheta_w = \sum_{k=1}^{n} a_k \sigma^k,$$

then the dimensionless heat flow may be written in the following way:

$$\frac{q}{q_{qu}} = \left(\frac{\sigma}{\sigma_a}\right)^{-1/2} \frac{1}{\vartheta_w}\left[\sum_{k=1}^{n} (-1)^{k+1} \frac{2k}{(2k-1)\cdot k!}\, \sigma^k \vartheta_w^{(k)}\right];\quad \sigma \leqslant \sigma_a, \tag{17a}$$

$$\frac{q}{q_{qu}} = 1 + \frac{1}{\vartheta_w}\left[\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{(2k-1)\cdot k!}\, \sigma_a^k \vartheta_w^{(k)}\right];\quad \sigma \geqslant \sigma_a, \tag{17b}$$

where q_{qu} is the quasistationary heat flow determined by Eq. (11b); ϑ_w^k is the k-th derivative of ϑ_w with respect to σ.

Figure 3 represents Eqs. (17a) and (17b) for the Prandtl number Pr = 0.72 in the case in which the surface temperature has the form:

$$\vartheta_w = a_n \sigma^n,$$

where n = 0, 1, 2, 3, 5, 7, and 10 (n = 0 is used in (11)).

We see from Fig. 3 that, the greater the degree n, i.e., the more sharply the surface temperature varies with time, the stronger is the heat flow.

The third and last case which we shall consider here is that in which for $\tau < 0$ there is a state of stationary heat transfer, i.e., $\vartheta_w = T_w - T_\infty = \mathrm{const} \neq 0$ for $\tau < 0$, and then at $\tau = 0$ the surface temperatures starts varying arbitrarily with time, i.e., $\vartheta_w = \vartheta_w(\tau)$ for $\tau > 0$.

Let us suppose that, when the surface temperature of the plates changes, the thickness of the thermal layer remains constant, i.e., $\delta(x, \tau) = \delta(x)$, and only the temperature field in the thermal boundary layer varies, i.e., $\vartheta = \vartheta(x, y, \tau)$ for $\tau > 0$.

Let the temperature profile be described by the equation of a cubic parabola

$$\vartheta = a_0(x, \tau) + a_1(x, \tau)\, y + a_2(x, \tau)\, y^2 + a_3(x, \tau)\, y^3. \tag{18}$$

The boundary conditions are:

$$\vartheta(x, \delta, \tau) = 0; \tag{19}$$

$$\frac{\partial \vartheta}{\partial y}(x, \delta, \tau) = 0; \tag{20}$$

$$\vartheta(x, 0, \tau) = \vartheta_w(\tau); \tag{21}$$

$$\frac{\partial^2 \vartheta}{\partial y^2}(x, 0, \tau) = \frac{1}{a}\frac{\partial \vartheta}{\partial \tau}(x, 0, \tau) = \frac{1}{a}\vartheta_w'(\tau). \tag{22}$$

The last condition (22) is obtained from Eq. (1) on substituting u = v = 0 into the latter. Using conditions (19)-(22) we obtain the following temperature profile:

$$\vartheta = \vartheta_w(\tau)\left[1 - \frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] + \frac{\delta^2}{a}\vartheta_w'(\tau)\left[\frac{1}{4}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2 + \frac{1}{4}\left(\frac{y}{\delta}\right)^3\right]. \tag{23}$$

The heat flow is determined by the Fourier law (9). Using Eq. (23) we find

$$q = \frac{3}{2}\frac{\lambda\vartheta_w(\tau)}{\delta} + \frac{1}{4}\frac{\gamma\delta\vartheta_w'(\tau)}{a}. \quad (24)$$

If $\vartheta_W' = 0$, i.e., ϑ_W = const, then q is given by Eq. (10). Hence Eq. (24) may be rewritten as:

$$\frac{q}{q_{qu}} = 1 + \frac{\delta^2}{6a}\frac{\vartheta_w'(\tau)}{\vartheta_w(\tau)}. \quad (25)$$

The thickness of the boundary layer $\delta(x)$ for stationary heat transfer is given by Eq. (8b); then on using Eq. (14) Eq. (25) takes the form

$$\frac{q}{q_{qu}} = 1 + 3.41\,\mathrm{Pr}^{1/3}\left(\frac{x}{U}\right)\frac{\vartheta_w'(\tau)}{\vartheta_w(\tau)} \quad (26a)$$

or with the aid of (15)

$$\frac{q}{q_{qu}} = 1 + 1.33\sigma_a\frac{\vartheta_w'(\sigma)}{\vartheta_w(\sigma)}. \quad (26b)$$

We see from Eq. (26) that, if $|\vartheta_W|$ increases with time, i.e., ϑ_W and ϑ_W' have the same sign, then the second term on the right-hand side of (26) will always be positive, and hence the ratio of the heat flow q/q_{qu} will be greater than unity.

NOTATION

T	is the temperature of the liquid in the thermal boundary layer;
T_∞	is the temperature of the flow of liquid at infinity;
T_W	is the surface temperature of the plate;
$\vartheta = T - T_\infty$;	
$\vartheta_W = T_W - T_\infty$;	
τ	is the time;
x	is the linear coordinate reckoned along the plate from the leading edge;
y	is the linear coordinate reckoned along the normal to the plate;
u	is the velocity component along the x axis;
v	is the velocity component along the y axis;
U	is the velocity of the flow of liquid;
a	is the thermal diffusivity;
ν	is the kinematic viscosity;
δ	is the thickness of thermal boundary layer;
$\varphi = \delta^2$;	
q	is the thermal flux density;
λ	is the thermal conductivity;
α	is the heat-transfer coefficient;
$\mathrm{Pr} = \nu/\alpha$	is the Prandtl number;
$\mathrm{Nu}_x = \alpha x/\lambda$	is the local Nusselt number;
$\mathrm{Re}x = U_x/\nu$	is the local Reynolds number;
$\sigma = U\tau/x$;	
σ_a	is the transition time given by Eq. (15);
q_{qu}	is the quasistationary (steady-state) thermal flux density;
s	is the variable in Laplace transformations;
t	is the integration variable.

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